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CYCLIC VECTORS IN FOCK-TYPE SPACES

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1. Introduction

Let \mathbb{D} be the open unit disk of the complex plane \mathbb{C} . We denote the polynomial ring by \mathcal{C} , and the space of all entire functions by $Hol(\mathbb{C})$. Let X be a complete semi-normed space of holomorphic functions on a domain Ω in \mathbb{C} . For a subset E of X , let \overline{E} be the closure of E in X . A function f is said to be *cyclic* in X if $f\mathcal{C} \subset X$ and $\overline{f\mathcal{C}} = X$. In the Hardy spaces $H^p(\mathbb{D})$ ($0 < p < \infty$), it is well known that a function is cyclic if and only if it is $H^p(\mathbb{D})$ -outer (see [Gar]). Also in the Bergman spaces $L_a^p(\mathbb{D})$ ($0 < p < \infty$), it is known that a function is cyclic if and only if it is $L_a^p(\mathbb{D})$ -outer (see [HKZ]). Recently the author has characterized the cyclic vectors in the classical Fock space. The classical Fock space $L_a^2(\mathbb{C})$ is

$$L_a^2(\mathbb{C}) = \left\{ f \in Hol(\mathbb{C}) : \|f\|_{L_a^2(\mathbb{C})} = \left\{ \int_{\mathbb{C}} |f(z)|^2 d\mu(z) \right\}^{1/2} < \infty \right\}$$

where

$$d\mu(z) = e^{-\frac{|z|^2}{2}} dA(z)/2\pi$$

is the Gaussian measure on \mathbb{C} and dA is the ordinary Lebesgue measure. In [Izu1], we have proved the following:

Theorem A . Let $h(z) \in Hol(\mathbb{C})$. Then the following are equivalent:

- (i) $f(z)$ is a nonvanishing function in $L_a^2(\mathbb{C})$.
- (ii) $f(z) = e^{h(z)}$, $h(z) = \alpha z^2 + \beta z + \gamma$ for $\alpha, \beta, \gamma \in \mathbb{C}$, $|\alpha| < \frac{1}{4}$.
- (iii) $f(z)$ is cyclic in $L_a^2(\mathbb{C})$.

It is known that there are non-vanishing functions in $H^p(\mathbb{D})$ and $L_a^p(\mathbb{D})$ which are not cyclic in the respective spaces (see [Gar] and [HKZ]). In fact, Brown and Shields posed the following question [BS]:

Question B . Let Ω be bounded region in \mathbb{C} . Does there exist a polynomially dense Banach space X of analytic functions in Ω with the two properties

- (i) $zX \subset X$
 - (ii) for any $\lambda \in \Omega$, point evaluation functional for λ is bounded,
- in which a function $f(z)$ is cyclic if and only if $f(z) \neq 0$ for all $z \in \Omega$?

The above theorem is not the answer of this question. But it says that there exists a polynomially dense Banach space in which every non-vanishing function is cyclic.

In this paper, we consider the cyclic vectors in more generalized spaces.

Let $0 < p < \infty$, $s > 0$ and $\alpha > 0$. Let ϕ be a positive function on $[0, \infty)$. The space $L_a^p(\mathbb{C}, \phi)$ consists of those entire functions whose semi-norm

$$\|f\|_{L_a^p(\mathbb{C}, \phi)} = \left\{ \frac{1}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(|z|)} dA(z) \right\}^{1/p}$$

is finite. This space is called Fock-type space. Throughout this paper, we put $\phi(|z|) = \frac{\alpha}{p}|z|^s$. We study the cyclic vectors in $L_a^p(\mathbb{C}, \phi)$.

This is a summary of the paper [Izu2].

2. Results

The following is our main result:

Theorem 1. *Let f be a function in $L_a^p(\mathbb{C}, \phi)$ satisfying $f\mathcal{C} \subset L_a^p(\mathbb{C}, \phi)$. Then the following are equivalent:*

- (i) $f(z)$ is a non-vanishing function.
- (ii) $f(z) = e^{h(z)}$ for $h(z) = \sum_{k=0}^{[s]} a_k z^k$, $a_k \in \mathbb{C}$, where $[s]$ is the largest integer with $[s] \leq s$, and in addition $|a_s| < \frac{\alpha}{p}$ if s is an integer.
- (iii) $f(z)$ is cyclic in $L_a^p(\mathbb{C}, \phi)$.

We know that every non-vanishing function in the classical Fock space $L_a^2(\mathbb{C})$ is cyclic. In our case, we notice that it is not valid for some positive numbers; that is, if s is not an integer or $s = 1, 2, 3, 4$, then $L_a^p(\mathbb{C}, \phi)$ has the same property as the one in $L_a^2(\mathbb{C})$, but if $s = 5, 6, 7, \dots$, the situation is different. For example, although $f(z) = e^{\frac{\alpha}{p}z^s}$ is a non-vanishing function in $L_a^p(\mathbb{C}, \phi)$, the function $f(z)$ does not satisfy $f\mathcal{C} \subset L_a^p(\mathbb{C}, \phi)$. Obviously this function $f(z)$ is not cyclic. But if we consider the non-vanishing functions just satisfying $f\mathcal{C} \subset L_a^p(\mathbb{C}, \phi)$, then the situation is similar.

To prove Theorem 1, we introduce the space \mathcal{F}_ϕ^p which is studied in [MMO]. The space is

$$\mathcal{F}_\phi^p = \left\{ f \in \text{Hol}(\mathbb{C}) : \|f\|_{\mathcal{F}_\phi^p}^p = \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(|z|)} \rho^{-1} \Delta\phi dA(z) < \infty \right\}$$

where $\Delta\phi$ is the Laplacian of ϕ and $\rho^{-1}\Delta\phi$ is a regular version of $\Delta\phi$. If $p = 2$, then \mathcal{F}_ϕ^2 is a Hilbert space with inner product

$$\langle f, g \rangle_{\mathcal{F}_\phi^2} = \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-2\phi(z)} \rho^{-1} \Delta\phi dA(z).$$

We denote the reproducing kernel of \mathcal{F}_ϕ^2 by K_λ , $\lambda \in \mathbb{C}$. The following lemma is proved by Marco, Massaneda and Ortega-Cerdà in [MMO, Lemma 21].

Lemma 2. *There exists a positive number C such that for any $\lambda \in \mathbb{C}$*

$$C^{-1} e^{2\phi(\lambda)} \leq \|K_\lambda\|_{\mathcal{F}_\phi^2}^2 \leq C e^{2\phi(\lambda)}.$$

In [CGH], Chen, Guo and Hou proved the following:

Lemma 3.

$$\lim_{|\lambda| \rightarrow \infty} \frac{\langle f, K_\lambda \rangle_{\mathcal{F}_\phi^2}}{\|K_\lambda\|_{\mathcal{F}_\phi^2}} = 0$$

for any $f \in \mathcal{F}_\phi^2$.

By Lemma 2 and 3, we get the following:

Lemma 4. *The following are equivalent:*

- (i) $f(z) \in L_a^p(\mathbb{C}, \phi)$ is a non-vanishing function satisfying $f\mathcal{C} \subset L_a^p(\mathbb{C}, \phi)$.
- (ii) $f(z) = e^{h(z)}$ where $h(z) = \sum_{k=0}^{[s]} a_k z^k$ and in addition $|a_s| < \frac{\alpha}{p}$ if s is an integer.

By Lemma 4, (i) \Leftrightarrow (ii) in Theorem 1 has been proved.

The following two lemmas are the generalizations of the results in [GW].

Lemma 5. *Let $f \in L_a^p(\mathbb{C}, \phi)$ with $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Then we have the following:*

- (i) *There exists a constant $C_1 > 0$, which depends on f , satisfying*

$$|c_n| \leq C_1 e^{\frac{2-s}{ps}} \left(\frac{s\alpha e}{pn + 2 - s} \right)^{\frac{n}{s}} \|f\|_{L_a^p(\mathbb{C}, \phi)}.$$

(ii) For large n ,

$$\begin{aligned}\|z^n\|_{L_a^p(\mathbb{C}, \phi)}^p &= \frac{\alpha^{-\frac{pn+2}{s}}}{s} \Gamma\left(\frac{pn+2}{s}\right) \\ &\sim \frac{1}{s\alpha} \left(2\pi \frac{pn+2-s}{s}\right)^{\frac{1}{2}} \left(\frac{pn+2-s}{s\alpha e}\right)^{\frac{pn+2-s}{s}},\end{aligned}$$

where Γ denotes the gamma function.

(iii) There is a constant $C_2 > 0$, which depends on f , satisfying

$$\|c_n z^n\|_{L_a^p(\mathbb{C}, \phi)} \leq C_2 \left(\frac{pn+2-s}{s}\right)^{\frac{1}{2p}} \left(\frac{pn+2-s}{s\alpha}\right)^{\frac{2-s}{ps}} \|f\|_{L_a^p(\mathbb{C}, \phi)}.$$

Using Lemma 5, we get the following lemma:

Lemma 6. *The polynomial ring \mathcal{C} is dense in $L_a^p(\mathbb{C}, \phi)$.*

Finally we show (ii) \Leftrightarrow (iii) in Theorem 1. Since every cyclic vector is non-vanishing, (iii) \Rightarrow (ii) is trivial. The idea for proving the opposite direction is from [Izu1].

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